Dynamics of ultraslow optical solitons in a cold three-state atomic system

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We present a systematic study on the dynamics of a ultraslow optical soliton in a cold, highly resonant three-state atomic system under Raman excitation. Using a method of multiple scales we derive a modified nonlinear Schrödinger equation with high-order corrections that describe effects of linear and differential absorption, nonlinear dispersion, delay response of nonlinear refractive index, diffraction, and third-order dispersion. Taking these effects as perturbations we investigate in detail the evolution of the ultraslow optical soliton using a standard soliton perturbation theory. We show that due to these high-order corrections the ultraslow optical soliton undergoes deformation, change of propagating velocity, and shift of oscillating frequency. In addition, a small radiation superposed by dispersive waves is also generated from the soliton. The results of the present work may provide a guidance that is useful for experimental demonstration of ultraslow optical soliton in cold atomic systems.

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I. INTRODUCTION

Solitons describe a class of fascinating shaping-preserving wave propagation phenomena in nonlinear media [1,2]. These special types of wave packets are formed as the result of interplay between nonlinearity and dispersion properties of the medium under excitations, and can lead to undistorted propagation over extended distance. Solitons have been discovered in many branches of physics and states of matters ranging from solid medium, such as optical fiber (optical soliton [3,4]), to Bose-Einstein condensed atomic vapor (matter wave solitons [5,6]).

Among various solitons studied so far, optical solitons are most extensively investigated because of the potential applications in information processing and transmission [3,4]. Due to the far-off-resonance nature of the excitation schemes, the conventional methods of generating optical solitons require intense electromagnetic radiation, and typically are generated with ultrashort pulsed laser systems. As a consequence of such strong excitations, optical solitons produced in the conventional methods generally travel with a speed very closed to the speed of light in vacuum and require media of extended lengths. Such high propagation speeds and requirements of extended propagation distances pose some significant obstacles for certain wave propagation devices which must have small form factor before being implemented into existing telecom systems.

In the past 10 years, research in the field of optical physics has seen a significant surge of activities using highly resonant media and index manipulation techniques such as the electromagnetically induced transparency (EIT) [7] technique. Some striking wave propagation effects [8–11] in such a highly resonant optical medium have been predicted and observed. One of the most noticeable cases is the significant modification of medium dispersion properties and the corresponding effect of significant reduction of the group velocity of an optical wave packet [12,13] under weak driving conditions. It has been shown that such ultraslow propagation of optical fields can lead to several new physical effects that are important from both fundamental physics and technology view points [14,15]. This is one of the reason ultraslow propagation of optical waves have been vigorously pursued in both fields of fundamental research and technological development.

Recently, highly resonant media and index manipulation technique have been studied for the possibility of generating shape-preserving propagations in both three-level [16] and four-level [17] media. Deng *et al.* reported a new type of optical solitons ultraslow optical solitons, that have drastically different formation and propagation characteristics and dynamics in comparison with the optical solitons generated with intense laser fields in optical fibers. In the conventional methods of optical soliton generation, the medium is excited by far-off resonance excitation fields. Such far-off-resonance excitations necessarily imply that the nonlinear effects are weak and extensive propagation distances are required to reach the balance between the nonlinearity and the dispersions of the medium. In a highly resonant medium, the excitation fields are very close to the resonances, thus a weak excitation field can induce large nonlinearity in the medium. This is particularly important because in a highly resonant scheme such as a three-state EIT scheme ultraslow propagation can be achieved only with reduced driving field. Such weak driving conditions, however, lead to significant probe field attenuation and spreading. In order to reduce these detrimental dispersion effects, large nonlinearity in a short propagation distance is required.

In this work we present a detailed study of the dynamics of an ultra slow optical soliton in a cold, highly resonant three-state atomic system. Using a method of multiple scales we first derive high-order corrections of the NLS equation including effects of linear and differential absorption, nonlinear dispersion, delay in nonlinear refractive index, and third-order dispersion. Using a standard soliton perturbation theory, we then investigate the formation, evolution and dynamics of the ultraslow optical soliton. The paper is arranged as follows. In Sec. II we describe a three-state model and discuss the solution in the linear regime. In Sec. III we give

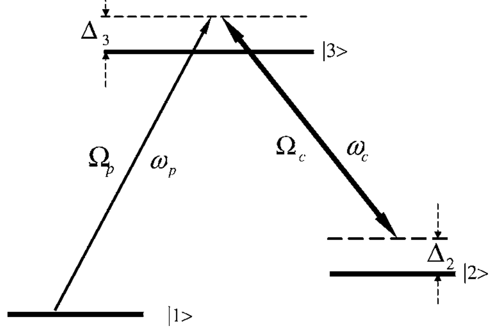

FIG. 1. Lifetime-broadened three-state atomic system interacting with a strong continuous-wave controlling field (with angular frequency ω_c and Rabi frequency $2\Omega_c$) and a pulsed probe field (with angular frequency ω_p and Rabi frequency $2\Omega_p$).

a detailed derivation of a modified NLS equation describing the evolution of an optical field. In Sec. IV we provide the soliton solution and investigate the formation and propagation dynamics of ultraslow solitons. In Sec. V discussions and comparisons with ultraslow optical solitons reported are presented. In Sec. VI, a summary of the present work is given.

II. MODEL AND SOLUTION IN LINEAR REGIME

We consider a lifetime broadened three-state atomic system that interacts with a weak, pulsed (pulse length τ_0) probe field of center frequency $\nu_p=\omega_p/(2\pi)$ ($|1\rangle\rightarrow|3\rangle$ transition) and a strong, continuous-wave (cw) control field of frequency $\nu_c=\omega_c/(2\pi)$ ($|2\rangle\rightarrow|3\rangle$ transition), respectively (Fig. 1). We assume that the cw control field is strong and not depleted during propagation. In interaction picture the equations of motion for the atomic state amplitudes and probe field can be written as

$$\left(i\frac{\partial}{\partial t}+d_2\right)A_2+\Omega_c^*A_3=0, \tag{1a}$$

$$\left(i\frac{\partial}{\partial t}+d_3\right)A_3+\Omega_pA_1+\Omega_cA_2=0, \tag{1b}$$

$$|A_1|^2+|A_2|^2+|A_3|^2=1, \tag{1c}$$

$$i\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\Omega_p+\frac{c}{2\omega_p}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Omega_p+\kappa A_3A_1^*=0, \tag{1d}$$

where A_j is the probability amplitude of the state $|j\rangle$ ($j=1,2,3$), $d_2=\Delta_2+i\gamma_2$ with Δ_2 being the two-photon detuning between the states $|1\rangle$ and $|2\rangle$ and γ_2 the decay rate of the state $|2\rangle$, and $d_3=\Delta_3+i\gamma_3$ with Δ_3 being the one-photon detuning between the states $|1\rangle$ and $|3\rangle$ and γ_3 the decay rate of the state $|3\rangle$. In addition, $2\Omega_p$ and $2\Omega_c$ are Rabi frequencies of the probe and control fields for the relevant transitions, and $\kappa=2\pi N\omega_p|D_0|^2/(\hbar c)$ where N is the concentration and D_0 is the dipole moment for the transition from $|1\rangle\rightarrow|3\rangle$.

Our objective is to demonstrate the interplay and balance between the dispersion effect and nonlinearity of the system under excitation. This is the key for the formation of shape-preserving propagation such as a soliton. In order to achieve this, we first examine the linear dispersion properties of the atomic system. We will assume that the probe field is weak that the ground state is not depleted. Thus, we take $A_1\approx 1$. The general procedures for calculating various dispersion effects is to first seek the leading order contributions for A_3 and use the result to construct the polarization term in the Maxwell equation (1d). With the assumption of nondepleted ground state and control field, Eqs. (1a)–(1c) can be solved using the standard Fourier transform method. This gives the Fourier transform of A_3. It is then straightforward to integrate Eq. (1d) along the propagation axis, yielding the solution of the probe field in the Fourier space. Finally, we carry out inverse transform to formally obtain the probe fieldmaly obtain the probe field as

$$\Omega_p(z,t)=\int_{-\infty}^{\infty}d\omega F(\omega)\exp\{i[K(\omega)z-\omega t]\},$$

where $F(\omega)=(1/2\pi)\int_{-\infty}^{\infty}dt\Omega_p(0,t)\exp(i\omega t)$ with $\Omega_p(0,t)$ being the probe field at the entrance of the medium. The dispersion function can be easily found as

$$K(\omega)=\frac{\omega}{c}+\frac{\kappa(\omega+d_2)}{D}, \tag{2}$$

where $D=D(\omega)=|\Omega_c|^2-(\omega+d_2)(\omega+d_3)$. In obtaining Eq. (2) we have neglected the transverse diffraction effect which, for the leading order, is usually very small. In most operation conditions $K(\omega)$ can be expanded into a rapidly conversion power series around the center frequency ω_p of the probe field, that is, $\omega=0$. We thus have

$$K(\omega)=K_0+K_1\omega+\frac{1}{2}K_2\omega^2+\cdots,$$

where $K_j=[\partial^jK(\omega)/\partial\omega^j]|_{\omega=0}$ ($j=0,1,2,\ldots$). These dispersion coefficients can be obtained analytically. For a Gaussian input of the probe field we have $\Omega_p(0,t)=\Omega_p(0,0)\exp(-t^2/\tau_0^2)$. Substituting this initial pulse profile into the expression of $F(\omega)$ and keeping terms up to ω^2, we obtain [18,19]

$$\Omega_p(z,t)=\frac{\Omega_p(0,0)}{\sqrt{b_1(z)-ib_2(z)}}\exp\left[iK_0z-\frac{(K_1z-t)^2}{[b_1(z)-ib_2(z)]\tau_0^2}\right], \tag{3}$$

where $b_1(z)=1+2z\,\mathrm{Im}(K_2)/\tau_0^2$ and $b_2(z)=2z\,\mathrm{Re}(K_2)/\tau_0^2$. Equation (3) clearly shows that linear and quadratic dispersion effects contribute to the probe field attenuation, phase shift, group velocity, and propagation-dependent pulse spreading. In the case of a Gaussian input pulse, it has been shown [18,19] that K_0 and K_1 scale as $|\Omega_c\tau_0|^{-2}$, and K_2 scales as $|\Omega_c\tau_0|^{-4}$. Thus, these dispersion coefficients, especially the pulse spreading, increase rapidly as the Ω_c is reduced significant for the purpose of achieving ultra slow group velocity. The key to possible shape-preserving propagation is to find effective remedy to balance the rapid increase pulse width in the time domain. This is the main objective of the next section where a nonlinear Schrödinger equation is derived and self-phase modulation effect is investigated for balancing the pulse spreading.

III. ASYMPTOTIC EXPANSION AND MODIFIED NLS EQUATION

In this section, we apply a weak nonlinear perturbation theory to a Raman scheme [18,20] with nonvanishing one- and two-photon detunings and search for stable soliton formation and propagation. The key reason for introducing a nonvanishing two-photon detunings is the requirement of preserving the integrability of a nonlinear Schrödinger equation governing the formation and propagation of solitons. Indeed, it is necessary to have $|\Delta_2| \gg \gamma_2$ so that the evolution equation for the probe field can be reduced to a nonlinear Schrödinger equation (at least to the leading order) which admits stable soliton type of solutions.

To achieve the goal of balancing the detrimental dispersion effects described in the previous section, we investigate the leading effect due to weak Kerr-type of nonlinear contributions such as self-phase modulation. There are ample evidences that Kerr-type of contribution becomes important under weak driving conditions [10]. To be able to determine the leading nonlinear contributions, we must consider corrections to the undepleted state approximation. Thus, we will allow a small depletion of the ground state state due to a weak nonlinear effect.

As a first step for getting a quantitative description for the formation and dynamics of an ultraslow optical soliton we now derive the nonlinear envelope equation describing the evolution of the probe field by using a standard multiscale approach [21]. We start by making the following asymptotic expansion ($j=1,2,3$):

$$A_j = \sum_{n=0}^{\infty} \varepsilon^n a_j^{(n)}, \quad \Omega_p = \sum_{n=1}^{\infty} \varepsilon^n \Omega_p^{(n)}, \tag{4}$$

where ε is a small parameter characterizing the small population depletion of the ground state. Substituting Eq. (4) into Eq. (1c), we obtain the lowest order solutions $a_1^{(0)}=1$ and $a_1^{(1)}=a_2^{(0)}=a_3^{(0)}=0$.

To obtain a divergence-free expansion, all quantities on the right hand side (RHS) of Eq. (4) are considered as functions of the multiscale variables $z_l=\varepsilon^l z$ ($l=0$ to 3), $t_l=\varepsilon^l t$ ($l=0,1$), $x_1=\varepsilon x$, and $y_1=\varepsilon y$. Using Eq. (4) and the multiscale variables given above, we can reduce Eqs. (1a)–(1d) to ($l=1-4$)

$$\left(i\frac{\partial}{\partial t_0}+d_2\right)a_2^{(l)}+\Omega_c^* a_3^{(l)}=M^{(l)}, \tag{5a}$$

$$\left(i\frac{\partial}{\partial t_0}+d_3\right)a_3^{(l)}+\Omega_p^{(l)}+\Omega_c a_2^{(l)}=N^{(l)}, \tag{5b}$$

$$i\left(\frac{\partial}{\partial z_0}+\frac{1}{c}\frac{\partial}{\partial t_0}\right)\Omega_p^{(l)}+\kappa a_3^{(l)}=Q^{(l)}. \tag{5c}$$

In deriving Eqs. (5a)–(5c) we have used Eq. (1c), i.e., $a_1^{(2)}+(a_1^{(2)})^*=-[a_2^{(1)}(a_2^{(1)})^*+a_3^{(1)}(a_3^{(1)})^*]$ and $a_1^{(3)}+(a_1^{(3)})^* =-[a_2^{(1)}(a_2^{(2)})^*+a_3^{(1)}(a_3^{(2)})^*+\text{c.c.}]$, where c.c. represents complex conjugate. The explicit expressions of $M^{(l)}$, $N^{(l)}$, and $Q^{(l)}$ can be systematically and analytically obtained.

To make the procedures of obtaining iterative solutions of Eqs. (5a)–(5c) more trackable, we express Eqs. (5a)–(5c) in the following forms:

$$a_2^{(l)}=\frac{1}{\Omega_c}\left[N^{(l)}-\left(i\frac{\partial}{\partial t_0}+d_3\right)a_3^{(l)}-\Omega_p^{(l)}\right], \tag{6a}$$

$$a_3^{(l)}=\frac{1}{\kappa}\left[Q^{(l)}-i\left(\frac{\partial}{\partial z_0}+\frac{1}{c}\frac{\partial}{\partial t_0}\right)\Omega_p^{(l)}\right], \tag{6b}$$

$$\hat{L}\Omega_p^{(l)}=S^{(l)}, \tag{6c}$$

where

$$\hat{L}=i\left(\frac{\partial}{\partial z_0}+\frac{1}{c}\frac{\partial}{\partial t_0}\right)\left[\left(i\frac{\partial}{\partial t_0}+d_2\right)\left(i\frac{\partial}{\partial t_0}+d_3\right)-|\Omega_c|^2\right] - \kappa\left(i\frac{\partial}{\partial t_0}+d_2\right),$$

$$S^{(l)}=\kappa\left[\Omega_c M^{(l)}-\left(i\frac{\partial}{\partial t_0}+d_2\right)N^{(l)}\right] + \left[\left(i\frac{\partial}{\partial t_0}+d_2\right)\left(i\frac{\partial}{\partial t_0}+d_3\right)-|\Omega_c|^2\right]Q^{(l)}.$$

Equations (6a)–(6c) can be solved order by order as shown below.

(i) *First-order approximation*. The case for $l=1$ is just the linear problem solved in the last section. We thus have the same linear dispersion relation (2) and the first-order approximation solutions ($j=2,3$)

$$\Omega_p^{(1)}=Fe^{i\theta}, \tag{7a}$$

$$a_j^{(1)}=\frac{-\delta_{j2}\Omega_c^*+\delta_{j3}(\omega+d_2)}{D}Fe^{i\theta}, \tag{7b}$$

where $\theta=K(\omega)z_0-\omega t_0=K(\omega)z-\omega t$ and δ_{ij} is Kronecker δ symbol. F is a yet to be determined envelope function depending on variables x_1, y_1, t_1 and z_j ($j=1,2,3$). Note that the dispersion function $K(\omega)$ enters the expression of the probe field, this implies that all order dispersion coefficients are appropriately included in the first-order approximation.

(ii) *Second-order approximation*. For $l=2$ we can obtain $M^{(2)}$, $N^{(2)}$, and $Q^{(2)}$ by substituting the first-order solution (7a) and (7b) into Eqs. (A1–A3), and also find the expression for $S^{(2)}$. Equation (6c) becomes

$$\hat{L}\Omega_p^{(2)}=iDe^{i\theta}\left(\frac{\partial F}{\partial z_1}+\frac{1}{V_g}\frac{\partial F}{\partial t_1}\right). \tag{8}$$

We note that since $\exp(i\theta)$ is the eigensolution of the operator $\hat{L}$, the solvability condition of Eq. (8) requires

$$i\left(\frac{\partial F}{\partial z_1}+\frac{1}{V_g}\frac{\partial F}{\partial t_1}\right)=0, \tag{9}$$

where $V_g=1/K_1$ is the group velocity of the wave packet. The second-order approximation solutions read

$$\Omega_p^{(2)}=0, \tag{10a}$$

$$a_1^{(2)}=-\frac{1}{2}\left(\left|\frac{\Omega_c}{D}\right|^2+\left|\frac{\omega+d_2}{D}\right|^2\right)|F|^2, \tag{10b}$$

$$a_2^{(2)}=-\frac{ie^{i\theta}}{\Omega_c\kappa}\left[\left(K-\frac{\omega}{c}\right)+(\omega+d_3)\left(\frac{1}{V_g}-\frac{1}{c}\right)\right]\frac{\partial F}{\partial t_1}, \tag{10c}$$

$$a_3^{(2)}=\frac{ie^{i\theta}}{\kappa}\left(\frac{1}{V_g}-\frac{1}{c}\right)\frac{\partial F}{\partial t_1}. \tag{10d}$$

(iii) *Third-order approximation*. This is the level where a nonlinear Schrödinger equation arises and the leading nonlinear contribution can have an effect in balancing the second order dispersion effect that causes pulse spreading. For $l=3$, we obtain

$$\hat{L}\Omega_p^{(3)}=D\left[i\frac{\partial F}{\partial z_2}-\frac{1}{2}K_2\frac{\partial^2 F}{\partial t_1^2}+\frac{c}{2\omega_p}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial y_1^2}\right)F-W|F|^2F\right]\times\exp(i\theta), \tag{11}$$

where

$$W=\frac{\kappa(\omega+d_2)(|\Omega_c|^2+|\omega+d_2|^2)}{D|D|^2}.$$

The solvability condition of the Eq. (11) gives rise to

$$i\frac{\partial F}{\partial z_2}-\frac{1}{2}K_2\frac{\partial^2 F}{\partial t_1^2}+\frac{c}{2\omega_p}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial y_1^2}\right)F-W|F|^2F=0, \tag{12}$$

which is the NLS equation describing the evolution of the envelope function F with the diffraction effect being taken into consideration. A similar equation without diffraction effects was also obtained in Ref. [16]. The integrability of Eq. (12) requires that the dominate part of W be real. This is the reason why one must choose nonvanishing two-photon detuning so that the dominate part in d_2 is Δ_2.

The above third-order approximation yields solutions

$$\Omega_p^{(3)}=0, \tag{13a}$$

$$a_1^{(3)}=iq_1F\frac{\partial F^*}{\partial t_1}, \tag{13b}$$

$$a_2^{(3)}=\left(p_1\frac{\partial^2 F}{\partial t_1^2}+p_2|F|^2F\right)\exp(i\theta), \tag{13c}$$

$$a_3^{(3)}=\left(-\frac{K_2}{2\kappa}\frac{\partial^2 F}{\partial t_1^2}+\sigma_3|F|^2F\right)\exp(i\theta), \tag{13d}$$

where

$$q_1=\frac{1}{D\kappa}\left[\left(K-\frac{\omega}{c}\right)+(\omega+d_3)\left(\frac{1}{V_g}-\frac{1}{c}\right)\right]^*+\frac{1}{\kappa}\left(K-\frac{\omega}{c}\right)\times\left(\frac{1}{V_g}-\frac{1}{c}\right)^*,$$

$$p_1=\frac{1}{\Omega_c\kappa}\left[\left(\frac{1}{V_g}-\frac{1}{c}\right)+\frac{1}{2}(\omega+d_3)K_2\right],\quad \text{(no number)}$$

$$p_2=\frac{\Omega_c^*}{2\kappa(\omega+d_2)}W,$$

$$\sigma_3=-\frac{W}{2\kappa}. \tag{14}$$

(iv) *The fourth-order approximation*. For $l=4$, from Eq. (6c) we have

$$\hat{L}\Omega_p^{(4)}=D(\omega)\left[i\frac{\partial F}{\partial z_3}-i\frac{K_3}{6}\frac{\partial^3 F}{\partial t_1^3}-i\beta_1\frac{\partial}{\partial t_1}(|F|^2F)+i\beta_2F\frac{\partial}{\partial t_1}(|F|^2)\right]\exp(i\theta), \tag{15}$$

where $K_3=\partial^3K/\partial\omega^3$ and

$$\beta_1=\frac{1}{2\kappa(\omega+d_2)D}\left[D^2W\left(\frac{1}{V_g}-\frac{1}{c}\right)+2\kappa^2(\omega+d_2)^2q_1\right],$$

$$\beta_2=\kappa\left[\frac{W}{4\kappa(K-\omega/c)}\left(\frac{1}{V_g}-\frac{1}{c}\right)+2q_1\frac{\omega+d_2}{D}\right].$$

The solvability condition of Eq. (15) results in

$$i\frac{\partial F}{\partial z_3}-i\frac{K_3}{6}\frac{\partial^3 F}{\partial t_1^3}-i\beta_1\frac{\partial}{\partial t_1}(|F|^2F)+i\beta_2F\frac{\partial}{\partial t_1}(|F|^2)=0. \tag{16}$$

Combining Eqs. (9), (12), and (16) we obtain

$$i\frac{\partial F}{\partial \zeta}-\frac{1}{2}K_2\frac{\partial^2 F}{\partial \eta^2}+\frac{c}{2\omega_p}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial y_1^2}\right)F-W|F|^2F+i\left[-\frac{K_3}{6}\frac{\partial^3 F}{\partial \eta^3}-\beta_1\frac{\partial}{\partial \eta}(|F|^2F)+\beta_2F\frac{\partial}{\partial \eta}(|F|^2)\right]=0, \tag{17}$$

where $\zeta=\varepsilon^2 z$ and $\eta=t_1-z_1/V_g$. Equation (17) is a modified NLS equation that has properly included the effects of diffraction, high-order dispersion, and high-order nonlinearity. Such an equation also appears in the study of soliton propagation in optical fibers [3,4].

IV. DYNAMICS OF ULTRASLOW OPTICAL SOLITONS

A. A preliminary discussion of Eq. (17)

The modified NLS equation (17) is a complex Ginzburg-Landau equation with high-order correction terms. Such an equation is generally not integrable and a soliton type of solution, even exists, may be highly unstable. However, if the imaginary part of each coefficient in Eq. (17) is small compared with the corresponding real part, a stable soliton solution may be supported for an extended propagation distance. This is, in general, the case involving solid state media such as optical fibers. In a cold atomic system, the effects due to imaginary parts of these coefficients may be reduced by appropriately choosing $\gamma_j\tau_0 \ll |\Delta_j\tau_0|$ $(j=1,2)$. The group velocity of the soliton wave packet will increase as the detunings are increased. Thus, a trade-off between the group velocity and the acceptable pulse profile distortion must be taken into consideration. Fortunately, a stable ultraslow soliton can form in a very short propagation distance. This is one of the significant advantages of the highly resonant system over the conventional optical fiber system, allowing direct manipulation techniques to be applied in the early stage of soliton formation. In the following, we will concentrate on this regime, and neglect most of decay constants. In the case of alkali metal vapor cooled down bellow Doppler limit, a detuning of Δ_2 of a few hundred kHz would be sufficient to validate the neglect of the decay rate of the state $|2\rangle$ yet still allows significantly slow group velocity. As pointed before, such a choice of coefficient is not possible with the conventional on one-photon resonance three-state EIT scheme under weak driving conditions, where $\Delta_2=0$ is always assumed.

Taking this into consideration and returning to original variables, Eq. (17) is replaced by

$$i\left(\frac{\partial}{\partial z}+\rho_1\right)U-\frac{1}{2}\widetilde{K}_2\frac{\partial^2 U}{\partial\tau^2}+\frac{c}{2\omega_p}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)U-\widetilde{W}|U|^2U + i\left[-\frac{\widetilde{K}_3}{6}\frac{\partial^3 U}{\partial\tau^3}-\widetilde{\beta}_1\frac{\partial}{\partial\tau}(|U|^2U)+\widetilde{\beta}_2U\frac{\partial}{\partial\tau}(|U|^2)\right]-\rho_2\frac{\partial U}{\partial\tau} = 0, \tag{18}$$

where we have taken $\omega=0$ and defined $\Omega_p=U\exp(i\widetilde{K}_0z)$, $\tau=t-z/\widetilde{V}_g$ with $\widetilde{K}_0=\kappa\Delta_2/(|\Omega_c|^2-\Delta_2\Delta_3)$. The real coefficients in Eq. (18) are given by

$$\widetilde{K}_2=\frac{2\kappa}{\widetilde{D}^2}\left[\Delta_2+\frac{(|\Omega_c|^2+\Delta_2^2)(\Delta_2+\Delta_3)}{\widetilde{D}}\right],$$

$$\widetilde{W}=\frac{\kappa\Delta_2[|\Omega_c|^2+\Delta_2^2]}{\widetilde{D}^3},$$

$$\widetilde{K}_3=\frac{6\kappa(|\Omega_c|^2+\Delta_2^2)}{\widetilde{D}^3}+\frac{3(\Delta_2+\Delta_3)}{\widetilde{D}}\widetilde{K}_2,$$

$$\widetilde{\beta}_1=\frac{1}{2\Delta_2\widetilde{D}}[(|\Omega_c|^2+\Delta_2^2)\widetilde{W}+\Delta_2^2\widetilde{K}_2],$$

$$\widetilde{\beta}_2=\frac{1}{2\Delta_2\widetilde{D}}\left[\frac{1}{2}(|\Omega_c|^2+\Delta_2^2)\widetilde{W}+2\Delta_2^2\widetilde{K}_2\right],$$

$$\rho_1=\frac{\kappa(|\Omega_c|^2\gamma_2+\Delta_2^2\gamma_3)}{\widetilde{D}^2},$$

$$\rho_2=\frac{2\kappa}{\widetilde{D}}\left[\frac{\Delta_2\gamma_2}{\widetilde{D}}+\frac{|\Omega_c|^2+\Delta_2^2}{\widetilde{D}^2}(\Delta_3\gamma_2+\Delta_2\gamma_3)\right],$$

with $\widetilde{D}=|\Omega_c|^2-\Delta_2\Delta_3$ and $\widetilde{V}_g=c/[1+\kappa c(|\Omega_c|^2+\Delta_2^2)/\widetilde{D}^2]$.

The key step in solving Eq. (18) is to analyze and estimate the relative importance of various terms in Eq. (18). To achieve this, we rewrite it into a dimensionless form using new variables $\sigma=\tau/\tau_0$ and $s=-z/(2L_D)$. In addition, we also rescale $(x,y)=R_\perp(x',y')$ and $U=U_0u$. With these new symbols, we have

$$i\frac{\partial u}{\partial s}+\frac{\partial^2 u}{\partial\sigma^2}+2|u|^2u = i\left[\bar{d}_0u-\bar{d}_1\frac{\partial(|u|^2u)}{\partial\sigma}+\bar{d}_2u\frac{\partial(|u|^2)}{\partial\sigma}-\bar{d}_3\frac{\partial^3u}{\partial\sigma^3}\right]-\bar{d}_4\frac{\partial u}{\partial\sigma} + \bar{d}_5\left(\frac{\partial^2u}{\partial x'^2}+\frac{\partial^2u}{\partial y'^2}\right), \tag{19}$$

where the dimensionless coefficients in Eq. (19) are given by $\bar{d}_j=2L_D/L_j$ $(j=0$ to $5)$, where $L_0=1/\rho_1$ (characteristic linear absorption length), $L_1=\tau_0/(\widetilde{\beta}_1U_0^2)$ (characteristic nonlinear dispersion length), $L_2=\tau_0/(\widetilde{\beta}_2U_0^2)$ (characteristic delay length in nonlinear refractive index [23]), $L_3=6\tau_0^3/\widetilde{K}_3$ (characteristic third-order dispersion length) $L_4=\tau_0/\rho_2$ (characteristic differential absorption [24] length), and $L_5=2\omega_pR_\perp/c$ (characteristic diffraction length), respectively. $L_D=\tau_0^2/\widetilde{K}_2$ is the characteristic dispersion length at which the group velocity dispersion becomes important. $R_\perp$ is beam radius of the probe field pulse. $L_{\rm NL}=1/(U_0^2\widetilde{W})$ is the effective length that characterize the influence of the nonlinearity. $2U_0=2(1/\tau_0)(\widetilde{K}_2/\widetilde{W})^{1/2}$ is a typical Rabi frequency of the probe field. We have assumed $L_D=L_{\rm NL}$, i.e., the balance of the dispersion and the nonlinearity, in order to favor the formation of soliton [we assume $\Delta_j>0$ $(j=2,3)$ so that both $\widetilde{K}_2$ and $\widetilde{W}$ are positive, corresponding to a bright soliton as discussed below [22]]. The physical meaning of each term in Eq. (19) can now be clearly stated. $\bar{d}_j$ $(j=0,\ldots,5)$ describes the significance of the various characteristic interaction lengths relative to the group velocity dispersion. The reason for choosing the group velocity dispersion as the metric for measuring these characteristic lengths is because it is the primary and leading source of distortion in wave propagation. These characteristic interaction lengths must be second order in comparison with the group velocity dispersion, by the choice of parameters, in order to achieve the desired balance using a Kerr-type of nonlinear interaction contribution alone, as discussed below.

B. Formation of ultraslow bright soliton

If L_j $(j=0,\ldots,5)$ are much longer than L_D, the terms on the right-hand side (RHS) of Eq. (19) are high-order ones and thus can be neglected within the propagating distance less than L_D. In this case Eq. (19) is reduced to the standard NLS equation

$$i\frac{\partial u}{\partial s}+\frac{\partial^2 u}{\partial \sigma^2}+2|u|^2u=0,$$

which is a completely integrable system and allows N-soliton solution [4,21]. The single soliton solution ($N=1$) is given by $u=\mathrm{sech}\,\sigma\exp(is)$, or in terms of field

$$\Omega_p = U\exp(i\tilde{K}_0 z) = \frac{1}{\tau_0}\left(\frac{\tilde{K}_2}{\tilde{W}}\right)^{1/2}\mathrm{sech}\left[\frac{1}{\tau_0}\left(t-\frac{z}{\tilde{V}_g}\right)\right]\exp\left[i\tilde{K}_0 z - i\frac{z}{2L_D}\right], \tag{20}$$

which describes a *bright* fundamental soliton traveling with propagating velocity $\tilde{V}_g$.

In the presence of the optical soliton the polarization of the atomic medium is given by

$$\wp = \frac{ND_0\Delta_2}{\tilde{D}}\frac{1}{\tau_0}\left(\frac{\tilde{K}_2}{\tilde{W}}\right)^{1/2}\mathrm{sech}\left[\frac{1}{\tau_0}\left(t-\frac{z}{\tilde{V}_g}\right)\right] \times\exp\left[i\tilde{K}_0 z - i\frac{z}{2L_D}\right]. \tag{21}$$

Thus, the atomic polarization wave is also a soliton type and travels with the same group velocity as the optical soliton.

We now present a numerical example using experimentally achievable parameters. We show that with appropriately selected parameters for a cold atomic system the RHS of Eq. (19) can indeed be considered as a small quantity. We consider a typical alkali system where the decay rates are $\Gamma_2=2\gamma_2=1.0\times10^4\ \mathrm{s}^{-1}$, $\Gamma_3=2\gamma_3=0.5\times10^7\ \mathrm{s}^{-1}$. We take $\kappa=1.0\times10^9\ \mathrm{cm}^{-1}\,\mathrm{s}^{-1}$, $2\Omega_c=2.4\times10^7\ \mathrm{s}^{-1}$, $\Delta_2=6.0\times10^5\ \mathrm{s}^{-1}$, $\Delta_3=5.0\times10^7\ \mathrm{s}^{-1}$, and $\tau_0=5.5\times10^{-6}\ \mathrm{s}^{-1}$, $\lambda_p=c/\nu_p=800$ nm, and $R_\perp=0.1$ cm. With these parameters we obtain $L_D=L_{\mathrm{NL}}=3.0$ cm, $L_0=8.0$ cm, $L_1=11.2$ cm, $L_2=11.9$ cm, $L_3=72.8$ cm, $L_4=16.1$ cm, and $L_5=1570$ cm. A bright soliton with spatial width $\tilde{V}_g\tau_0=0.49$ cm [i.e., Eq. (20)] forms at $z=3$ cm and the soliton propagates with group velocity of $\tilde{V}_g=3.0\times10^{-6}$ c/s. For a propagation distance of $z=3.5$ cm, various high-order effects remain small since $L_j \gg L_0=8$ cm ($j=0,\ldots,5$) and no significant degradation to the integrity of soliton propagation occurs. In this case the soliton solution, i.e., Eq. (20), is still a good approximate solution of Eq. (19). It is remakable that a stable ultraslow optical soliton can be formed in such a short propagation distance under such a weak driving conditions in a highly resonant medium.

We now discuss the input power requirement for forming stable ultraslow optical solitons in the parameter regime discussed in the previous subsection. The flux of energy of the probe optical field associated with a single soliton is given by Poynting vector integrated over the cross section of the sample:

$$P=\int dS(\mathbf{E}_p\times\mathbf{H}_p)\cdot\mathbf{e}_z,$$

where $\mathbf{e}_z$ is the unit vector in the propagation direction. To leading order the field is transverse and one has $\mathbf{E}_p=(E_p,0,0)$ then $\mathbf{H}_p=(0,H_p,0)$ with $H_p=\varepsilon_0 c n_p E_p$, where $n_p=1+c\tilde{K}/\omega_p$ is the refractive index of the probe field. Note that $E_p=(\hbar/D_0)\Omega_p\exp[i(\omega_p z/c-\omega_p t)]+\text{c.c.}$, one obtains the average flux of energy over carrier-wave period

$$\bar{P}=\bar{P}_{\max}\,\mathrm{sech}^2\left[\frac{1}{\tau_0}\left(t-\frac{z}{\tilde{V}_g}\right)\right]$$

with the peak power

$$\bar{P}_{\max}=2\varepsilon_0 c n_p S_0|E_p|^2_{\max}=2\varepsilon_0 c n_p S_0\left(\frac{\hbar}{D_0}\right)^2\frac{1}{\tau_0^2}\frac{\tilde{K}_2}{\tilde{W}},$$

where S_0 is the cross-section area of the probe laser beam. We see that the peak power is directly proportional to the dispersion coefficient $\tilde{K}_2$ and inversely proportional to the square of the pulse width τ_0 as well as the self-phase modulation coefficient $\tilde{W}$. Using the above numerical example and take $D_0=2.1\times10^{-27}$ cm C and $S_0=1.0\times10^{-2}\ \mathrm{cm}^2$, we obtain $|E_p|_{\max}=0.12V\ \mathrm{cm}^{-1}$ and $\bar{P}_{\max}=7.24\times10^{-4}$ mW. Thus, very low input power is needed for generating ultraslow optical soliton using highly resonant atomic medium. This is drastically different from the conventional optical soliton generation technique using optical fibers. In the latter case ultrashort pulses such as ps or fs laser pulses are necessary in order to reach sufficiently high peak power and therefore bring out the nonlinear effect required for soliton formation.

In general, the NLS equation admits an N-soliton solution. To generate a single soliton, one must prepare an appropriate initial condition. Note that the modulus area of the single-soliton envelope (20) is

$$A_t=\int_{-\infty}^{\infty}dt|\Omega_p(z,t)|=\pi\left(\frac{\tilde{K}_2}{\tilde{W}}\right)^{1/2}.$$

According to soliton theory [4,21], a single soliton will emerge out of any set of initial condition $\Omega(0,t)$ whose modulus area is between $A_t/2$ and $3A_t/2$.

C. Soliton deformation and radiation under linear absorption

In the regime where some of the higher order terms on the RHS of Eq. (19) must be included, new phenomena occur as the results of these contributions. These new effects include various radiation deformations and oscillations. To investigate some of these effects, we write the Eq. (19) into the following form:

$$i\frac{\partial u}{\partial s}+\frac{\partial^2 u}{\partial \sigma^2}+2|u|^2u=iR[u], \tag{22}$$

where $R[u]$ is a small quantity representing all high-order corrections and is taken as a perturbation the stable solution of Eq. (19). The procedure is to use the soliton solution of

Eq. (22) as the initial input to solve Eq. (22) for the given perturbation $R[u]$.

Equation (22) with $R[u]=0$ has the general four-parameter bright-soliton solution

$$u_0 = 2\beta\,\mathrm{sech}[2\beta(\sigma - \sigma_0 + 4\alpha s)] \times \exp[-2i\alpha\sigma - 4i(\alpha^2 - \beta^2)s - i\phi_0], \tag{23}$$

where α, β, σ_0, and ϕ_0 are real free parameters which determine the propagating velocity, amplitude (as well as width), initial position, and initial phase of the soliton, respectively. When the perturbation $R[u]\neq 0$ the soliton will undergo a deformation during propagation and generate a dispersive radiation. With $R[u]\neq 0$, soliton solution is given by $u=u_0+u_1$ [25,26], where

$$u_0 = 2\beta e^{-i\phi}\,\mathrm{sech}\, w, \tag{24a}$$

$$u_1 = e^{-i\phi}(A + iB), \tag{24b}$$

where

$$w = 2\beta(\sigma - \xi), \quad \phi = 2\alpha(\sigma - \xi) + \delta,$$

$$\frac{d\alpha}{ds} = -\frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty} dw R[u_0]e^{i\phi}\tanh w\,\mathrm{sech}\, w, \tag{25a}$$

$$\frac{d\beta}{ds} = \frac{1}{2}\,\mathrm{Re}\int_{-\infty}^{\infty} dw R[u_0]e^{i\phi}\,\mathrm{sech}\, w, \tag{25b}$$

$$\frac{d\xi}{ds} = -4\alpha + \frac{1}{4\beta^2}\,\mathrm{Re}\int_{-\infty}^{\infty} dw R[u_0]e^{i\phi} w\,\mathrm{sech}\, w, \tag{25c}$$

$$\frac{d\delta}{ds} = -4(\alpha^2 + \beta^2) + 2\alpha\frac{\partial\xi}{\partial s} - \frac{1}{2\beta}\mathrm{Im}\int_{-\infty}^{\infty} dw R[u_0]e^{i\phi} \times (1 - w\tanh w)\mathrm{sech}\, w, \tag{25d}$$

$$A = \int_{-\infty}^{\infty} dk\, a(s,k)\Psi(w,k), \quad B = \int_{-\infty}^{\infty} dk\, b(s,k)\Phi(w,k), \tag{25e}$$

and

$$a(s,k) = \frac{q(k)}{4\beta^2(k^2+1)}\{1 - \cos[4\beta^2(k^2+1)s]\} + \frac{p(k)}{4\beta^2(k^2+1)}\sin[4\beta^2(k^2+1)s],$$

$$b(s,k) = -\frac{p(k)}{4\beta^2(k^2+1)}\{1 - \cos[4\beta^2(k^2+1)s]\} + \frac{q(k)}{4\beta^2(k^2+1)}\sin[4\beta^2(k^2+1)s],$$

$$\Phi(w,k) = \frac{1}{\sqrt{2\pi}}(1 - k^2 - 2ik\tanh w)e^{ikw},$$

$$\Psi(w,k) = \frac{1}{\sqrt{2\pi}}\left(-1 - k^2 - 2ik\tanh w + 2\tanh^2 w\right)e^{ikw}, \tag{26}$$

where $p(k)=\int_{-\infty}^{\infty} dw\,\mathrm{Re}(R[u_0]e^{i\phi})\Phi^*(w,k)$ and $q(k)=\int_{-\infty}^{\infty} dw\,\mathrm{Im}(R[u_0]e^{i\phi})\Psi^*(w,k)$. The formulas (25a)–(25d) describe the adiabatic change of soliton parameters and u_1 is a dispersive wave radiated from the soliton due to the perturbation.

We first consider the linear absorption and the resulting corrections to the ultraslow optical soliton of form of Eq. (20), i.e., we take $R[u]=\bar{d}_0 u$. This approach is valid if the soliton propagates to distance much less than 8 cm. In this case the other high-order effects are insignificant and thus can be safely neglected. Using the above formulas we obtain $\alpha=\alpha_0$, $\beta=\beta_0\exp(2\bar{d}_0 s)$, $\xi=\xi_0-2\alpha_0 s$, and $\delta=\delta_0-4\alpha^2 s+\beta_0^2[1-\exp(4\bar{d}_0 s)]/\bar{d}_0$. Thus we have $u_0=2\beta_0\exp(2\bar{d}_0 s)\mathrm{sech}[2\beta_0\exp(2\bar{d}_0 s)(\sigma+4\alpha_0 s-\xi_0)]\exp(-i\phi)$ with $\phi=2\alpha_0(\sigma-\xi_0)+4\alpha_0^2 s+\beta_0^2[1-\exp(4\bar{d}_0 s)]/\bar{d}_0+\delta_0$, where α_0, β_0, ξ_0, and δ_0 are free parameters depending on initial condition. As a result, we obtain evolution of the ultraslow optical soliton (20) ($\alpha_0=\xi_0=\delta_0=0,\beta_0=1/2$) under the perturbation due to linear absorption

$$\Omega_p = \frac{e^{-\bar{d}_0 z/L_D}}{\tau_0}\left(\frac{\tilde{K}_2}{\tilde{W}}\right)^{1/2}\mathrm{sech}\left[\frac{e^{-\bar{d}_0 z/L_D}}{\tau_0}\left(t - \frac{z}{\tilde{V}_g}\right)\right] \times \exp\left[i\tilde{K}_0 z - i\frac{1 - e^{-2\bar{d}_0 z/L_D}}{4\bar{d}_0}\right]. \tag{27}$$

We see that the linear absorption results in a deformation of the soliton by decreasing the soliton amplitude and increasing the soliton width. It has, however, no effect on the propagating velocity of the soliton.

Next, we calculate the radiation part u_1. In the present case we have $q(k)=0$ and $p(k)=\sqrt{2\pi}\bar{d}_0\beta/\cosh(\pi k/2)$. Thus, we get $a(s,k)=\sqrt{2\pi}\bar{d}_0\sin[4\beta^2(k^2+1)s]/[4\beta(k^2+1)\cosh(\pi k/2)]$ and $b(s,k)=-\sqrt{2\pi}\bar{d}_0\{1-\cos[4\beta^2(k^2+1)s]\}/[4\beta(k^2+1)\cosh(\pi k/2)]$. Then A and B in Eq. (25e) are given by

$$A = \frac{\bar{d}_0}{4\beta}\int_{-\infty}^{\infty} dk e^{ikw}\frac{\sin[4\beta^2(k^2+1)z']}{(k^2+1)^2\cosh(\pi k/2)}[1 + k^2 + 2ik\tanh w - 2\tanh^2 w],$$

$$B = -\frac{\bar{d}_0}{4\beta}\int_{-\infty}^{\infty} dk e^{ikw}\frac{1 - \cos[4\beta^2(k^2+1)z']}{(k^2+1)^2\cosh(\pi k/2)} \times [1 - k^2 - 2ik\tanh w],$$

where $z'=z/(2L_D)$. Although the above integrals can be calculated by numerical methods, here we show an asymptotic solution in order to get a transparent physical insight. Using a method of steepest descents for large z and finite w/z' one can show the following integration formulas:

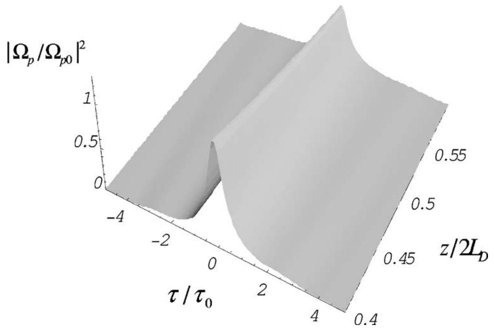

FIG. 2. Evolution of the relative probe field intensity $|\Omega_p/\Omega_{p0}|^2$ in the presence of linear absorption verse the dimensionless delay time $T=\tau/\tau_0$ and propagating distance $Z=z/(2L_D)$. The parameters are those given in Sec. IV B and $\beta_0=1/2$, $\alpha_0=\xi_0=\delta_0=0$.

$$\int_{-\infty}^{\infty} dkG_1(k)\exp[i(kw-4\beta^2k^2z')] \approx \frac{1}{2\beta}\sqrt{\frac{\pi}{z'}}G_1(k_0)\exp[i4\beta^2k_0^2z'-(i\pi/4)],$$

$$\int_{-\infty}^{\infty} dkG_2(k)\exp[i(kw+4\beta^2k^2z')] \approx \frac{1}{2\beta}\sqrt{\frac{\pi}{z'}}G_2(k_0)\exp[-i4\beta^2k_0^2z'+(i\pi/4)],$$

where $k_0=w/(8\beta^2z')$, $G_j(k)$ $(j=1,2)$ are analytic functions of k in certain region near the real axis. Applying the above formulas we get

$$A=\sqrt{\frac{\pi}{z'}}f(k_0)\{[(k_0^2+1)-2\tanh^2 w]\sin[4\beta^2(1-k_0^2)z'+\pi/4] - 2k_0\tanh w\cos[4\beta^2(1-k_0^2)z'+\pi/4]\}, \tag{28a}$$

$$B=-\frac{\bar{d}_0}{2\beta}(3-2\,\mathrm{sech}^2 w)\mathrm{sech}\, w+\sqrt{\frac{\pi}{z'}}f(k_0) \times\{(1-k_0^2)\cos[4\beta^2(1-k_0^2)z'+\pi/4] - 2k_0\tanh w\sin[4\beta^2(1-k_0^2)z'+\pi/4]\}, \tag{28b}$$

where $f(k_0)=d_0/[8\beta^2(k_0^2+1)^2\cosh(\pi k_0/2)]$. We see that A and the second term in B are continuous dispersive waves radiated from the soliton. The physical origin of such radiation is the soliton deforms caused by the perturbation. The perturbation causes the small deformation near the wing of the soliton wave packet where the optical intensity is weaker and the contraction due to the nonlinear interaction is much less than the center of the wave packet. Thus, the distortion on the wing breaks the balance between the dispersion and nonlinearity effects, causing rapid spreading at the wing. Consequently, the soliton loses part of its energy which is transferred into radiation energy. Note that the first term in B is a localized wave traveling with the soliton.

In Fig. 2 we have plotted $|\Omega_p/\Omega_{p0}|^2$ $[\Omega_{p0}\equiv\tau_0^{-1}(\widetilde{K}_2/\widetilde{W})^{1/2}]$ as a function of τ/τ_0 and $z/(2L_D)$ with the parameters given in Sec. IV B [27]. We see that because of linear absorption the soliton undergoes a small deformation. Its amplitude (width) decreases (increases). The propagation velocity, however, remains the same. In addition, small dispersive waves are radiated from the wing of the soliton due to the energy loss induced by the deformation.

D. Frequency shift and change of velocity due to high-order effects

We now consider other high-order terms in Eq. (19) and their effects on the ultraslow optical soliton. Since the third-order dispersion length (L_3) and the diffraction length (L_5) are much longer than other typical lengths (see Sec. IV B), the $\bar{d}_3$ and $\bar{d}_5$ terms on the RHS of Eq. (19) can be neglected safely. From Eqs. (25a)–(25d) one can easily obtain the equations of motion on the change of the soliton parameters

$$\frac{d\alpha}{dz'}=-\frac{4}{3}\bar{d}_4\beta^2,$$

$$\frac{d\beta}{dz'}=-2\bar{d}_0\beta-4\bar{d}_4\alpha\beta,$$

$$\frac{d\xi}{dz'}=4\alpha-\frac{4}{3}(3\bar{d}_1-2\bar{d}_2)\beta^2,$$

$$\frac{d\delta}{dz'}=4(\alpha^2+\beta^2)+\frac{16}{3}\bar{d}_2\alpha\beta^2,$$

where s has been replaced by z' $[z'=z/(2L_D)]$. As above we assume that the soliton [Eq. (20)] is input at $z=0$, then the initial condition of the above set of equations are $\beta_0=1/2$ and $\alpha_0=\xi_0=\delta_0=0$. Considering that $\bar{d}_4$ is much less than $\bar{d}_0$, the second equation above gives $\beta(z')=\exp(-2\bar{d}_0z')$. We thus obtain the analytical solution

$$\alpha(z')=\frac{\bar{d}_4}{12\bar{d}_0}(e^{-4\bar{d}_0z'}-1),$$

$$\xi(z')=-\frac{\bar{d}_4}{3\bar{d}_0}z'+\frac{1}{12\bar{d}_0}\left[3\bar{d}_1-\bar{d}_2-\frac{\bar{d}_4}{\bar{d}_0}\right](e^{-4\bar{d}_0z'}-1),$$

$$\delta(z')=\frac{1}{36}\left(\frac{\bar{d}_4}{\bar{d}_0}\right)^2z'-\frac{\bar{d}_4}{72\bar{d}_0^{\,2}}\left(\bar{d}_2+\frac{\bar{d}_4}{\bar{d}_0}\right)(e^{-8\bar{d}_0z'}-1)-\frac{1}{4\bar{d}_0} \times\left[1-\frac{\bar{d}_4}{18\bar{d}_0}\left(2\bar{d}_2+\frac{\bar{d}_4}{\bar{d}_0}\right)\right](e^{-4\bar{d}_0z'}-1). \tag{29}$$

Compare these with the expression of u_0 in Eq. (24a) we see that, in addition to the changes of the amplitude and width, the soliton also undergoes a variation of its oscillating frequency and and a change of propagation velocity. The shift of the oscillating frequency is given by

$$\Delta\omega_s = \frac{\bar{d}_4}{6\bar{d}_0\tau_0}(e^{-4\bar{d}_0 z'} - 1).$$

This shows that the differential absorption [represented by the $\bar{d}_4$ term in Eq. (19)] results in a down shift of the oscillating frequency of the soliton. For large z' the down shift reaches the maximum value of $\Delta\omega_{\max}=\bar{d}_4(6\bar{d}_0\tau_0)^{-1}=0.08\tau_0^{-1}$ for the parameters given in Sec. IV B.

It is easy to show that under the action of the perturbation the propagating velocity of the soliton changes into

$$V_s = \frac{\tilde{V}_g}{1 - \frac{\tilde{V}_g\tau_0}{6L_D}\left[\frac{\bar{d}_4}{\bar{d}_0} + \left(3\bar{d}_1 - \bar{d}_2 - \frac{\bar{d}_4}{\bar{d}_0}\right)e^{-4\bar{d}_0 z'}\right]}.$$

Therefore, nonlinear dispersion, delay in nonlinear refractive index, and the differential absorption, characterized by $\bar{d}_1$, $\bar{d}_2$, and $\bar{d}_4$ all contribute to the change of soliton propagation velocity. For the parameters given in Sec. IV B, $3\bar{d}_1-\bar{d}_2$ is positive and thus the soliton velocity increases comparing with the case without the high-order perturbation. For large z', V_s approaches the limit $\tilde{V}_g[1-\tilde{V}_g\tau_0\bar{d}_4/(6L_D\bar{d}_0)]^{-1}$, which takes the value $3.041\times10^{-6}c$ when using the numerical values of the parameters given in Sec. IV B. Thus, the soliton velocity change due to these high-order terms are negligibly small.

V. DISCUSSION AND AND COMPARISON WITH OTHER MODELS

The system described in the present study is a simple three-state scheme. In this system the origin of the nonlinearity that provides effective balances of the group velocity dispersion is the Kerr-type self-phase modulation effect of the probe field. This is very different from the ultraslow optical soliton reported previously where a four-state scheme and three laser fields were assumed. In the latter case, the origin of nonlinearity is the Kerr-type of cross-phase modulation involving a third (cw) laser field tuned far-off a fourth energy level. The advantages of the four-level system is its relaxed condition for achieving soliton propagation since both the detuning and intensity contribute to the process. Since the cross-phase modulation is provided by the second weak cw control field, the cross-phase modulation is nearly cw (depending on the location in the probe beam profile) and the instability in cross-phase modulation is reduced. In the case of a three-state system, the self-modulation effect is based on the probe field itself and thus pulsed in nature. This may introduce fluctuations and instabilities to the soliton formation and propagation. In addition, the four-state system allows a large intensity-dependent nonlinear phase shift to be added to the soliton wave packet. This may have applications in shaper-preserving nonlinear phase shifters. However, there are optoelectronic devices where phase shift must be avoided. In the latter applications, the three-state scheme is a more suitable candidate.

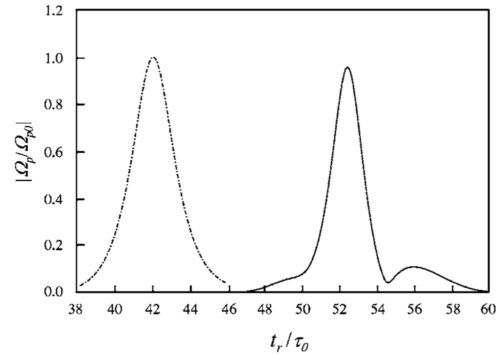

FIG. 3. Evolution of the relative probe field amplitude $|\Omega_p/\Omega_{p0}|$ obtained numerically without nondepleted ground state and constant control field approximations. The input field is assumed to have the form of Eq. (20). The soliton deformation and radiation on the wing can be seen clearly. However, the central peak preserves the shape and width of the initial input field remarkably well. Parameters used: $|\Omega_{p0}\tau|=6.3$, $|\Omega_c\tau|=40$, $\delta_2\tau=-40$, $\delta_3\tau=-2$, $\gamma_2\tau=3$, $\gamma_3\tau=0.0001$, $\kappa_{12}\tau=\kappa_{32}\tau=100\,000$, and $z=0.7$ cm. Dash-dotted line: soliton solution of Eq. (20). Solid line: full numerical solution of five differential equations with only the diffraction effect neglected. Notice the peak separation due to the group velocity modification, and radiation deformation on the wing, as predicted by the present theory.

From the theoretical study's view point, the present work is a significant development since the work reported previously. It is apparent that the methodology used in the previous studies are limited to the lowest order corrections in the sense of Ginzburg-Landau equation. Thus, it cannot predict effects such as soliton deformation, group velocity change, radiation energy loss, soliton central carrier frequency shift, and diffraction characteristics.

In Fig. 3, we show the relative probe field amplitude for a very different set of operation parameters. These parameters are chosen specifically to demonstrate soliton deformation, group velocity change, and radiation on the wing of the field profile. This figure is produced by numerically integrating all five differential equations, three for amplitudes of atomic wave function and two for amplitudes of the probe and control fields, without the nondepleted ground state and constant control field approximations. The only assumption is that the initial input field has a soliton profile very similar to that given in Eq. (20), and the diffraction effects are neglected. As can be seen, after about $z=0.7$ cm (or for about $t_r\tau_0\simeq 60$ time unit) the deformation has occurred on the wing and the the energy is radiated as the soliton propagates. However, the central peak is remarkably similar to the shape of the initial input field, although faster because of the group velocity modification, indicating the robust nature of the shape-preserving propagation of the optical soliton.

It should be pointed out that both the present and the previous studies are in the regimes where relative weak electromagnetically induced transparency occurs. This is necessary in order to achieve group velocity reduction. In principle, it is possible to produce optical solitons in other multilevel systems if the conditions for balancing the group dispersion can be realized [28,29].

It is also worth pointing out the differences in comparison with the optical solitons generated with the conventional methods. Aside from the ultraslow propagation velocity which may find applications in optical engineering, the schemes based on resonant media have some fundamental properties that are generally lacking in the conventional systems. For instance, it is possible to produce both dark and bright solitons with the same resonant medium by taking different operation conditions such as detunings. This is not possible in the conventional systems. The dispersion properties in the latter system is in general fixed and the entire medium must be changed in order to obtain optical solitons of different (as from bright to dark) characters. The key is that in the highly resonant medium one has the flexibility of adjusting and even choosing the sign of the interactions, leading to rich propagation phenomena not seen in optical fiber based systems.

VI. SUMMARY

In summary, in this work we have presented a systematic theoretical study on the dynamics of an ultraslow optical soliton in a highly resonant three-state atomic system. Using a method of multiple scales we have given a detailed derivation for a modified NLS equation with high-order corrections including effects of linear and differential absorption, nonlinear dispersion, delay in nonlinear refractive index, diffraction, and third-order linear dispersion. Using soliton perturbation theory we have studied physical properties of ultraslow optical soliton and investigated high-order effects on the soliton evolution in detail. The results show that due to the high-order corrections the ultraslow optical soliton undergoes deformation, change of propagating velocity, and shift of oscillating frequency. In addition, a small radiation superposed by dispersive waves is also shed from the soliton. However, as we have shown, the changes under the perturbation are small and hence the ultraslow optical soliton displays a robust nature during propagation. The results presented in this work may be useful in guiding experimental demonstration of the ultraslow optical soliton in cold atomic systems.

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[22] One can obtain dark solitons by changing the sign of Δ_j.
[23] The fourth term on the lhs of Eq. (18) is proportional to $(\omega_p/c)n_2|U|^2U$, where n_2 is Kerr coefficient. If it is replaced by $U\int_{-\infty}^{\infty}n_2(t-t')|U|^2(t')dt$, then after expansion it will contribute the term $u\partial(|u|^2)/\partial\sigma$ in Eq. (19). Thus we call the term proportional $\bar{d}_2$ as the term of delay in nonlinear refractive index.
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[27] In this work we do not discuss the formation process of the ultraslow optical soliton for a given initial condition. We consider only the propagation dynamics of the soliton after it has already formed. We assume the soliton forms at the position $z=0$. (This can always be done by translating the origin along the z axis.) and propagates along the positive z direction.
[28] Note that the ultraslow optical soliton discussed here is for long (probe) pulses with a weak light intensity. It is essentially different from the soliton under a mechanism of self-induced transparency in two-level atoms [S. L. McCall and E. L. Hahn, Phys. Rev. Lett. **18**, 908 (1967)], where a soliton forms under strong light and short pulse conditions and no EIT-like quantum interference effect is concerned.
[29] This can be seen from the expressions of K_2 [the second-order derivative of $K(\omega)$] and Eq. (12).